EFFECT OF CORIOLIS FORCES ON NONSTATIONARY INCOMPRESSIBLE VISCOUS LIQUID FLOW IN A CHANNEL

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We present a solution of the problem of nonstationary viscous incompressible liquid flow in a channel, taking account of the action of Coriolis forces. In this case the solution is shown to be determined by the sole dimensionless Taylor parameter, with an increase of which the velocity regime of nonstationary flow changes from a monotonic to a pulsating one.

The motion of a viscous incompressible liquid in noninertial coordinate systems exhibits a number of specific features, without account for which considerable errors may arise in both the description of the pattern of flow itself and calculation of its averaged hydrodynamic parameters.

Using as an example plane-parallel channel flow, we investigate the specific features of the effect of Coriolis forces on transition processes occurring in nonstationary incompressible viscous liquid flows. Problems of this kind in a stationary case were considered for the first time by Ekman to describe wind-induced oceanic flows on the rotating Earth [1, 2].

We will consider a flow of liquid moving between two infinite planes $y = \pm h$. We will consider the constant pressure gradient to be directed along the OZ axis. Then, for a plane nonstationary stabilized flow, in which the longitudinal w and transverse (parallel to the channel walls) u velocity components are independent of the x and z coordinates, the system of dimensionless differential equations, taking account of the action of Coriolis forces, can be written in the form [3]

$$\frac{\partial u}{\partial t} = \frac{1}{\mathrm{Ta}} \frac{\partial^2 u}{\partial y^2} - 2w \cos\beta ,$$

$$\mathrm{Eu} \frac{\mathrm{Re}}{\mathrm{Ta}} \frac{\partial P}{\partial y} + 2 (w \cos\alpha - u \cos\gamma) = 0 , \tag{1}$$

$$\frac{\partial w}{\partial t} = - \mathrm{Eu} \frac{\mathrm{Re}}{\mathrm{Ta}} \frac{\partial P}{\partial z} + \frac{1}{\mathrm{Ta}} \frac{\partial^2 w}{\partial y^2} + 2u \cos\beta .$$

The following model problem can be set to correspond with the system of equations (1). Suppose that up to a certain initial time instant the liquid was at rest. At that time instant $t = 0$, along the OZ axis, a constant pressure gradient appears which causes the motion of the liquid. Let us elucidate the special features of this transitional nonstationary process under conditions of the additional effect of Coriolis forces. The solution of a problem of this kind in an inertial coordinate system for the case of flow in a channel and in a round tube is given in [2, 4, 5].

Obviously, for the above problem there will correspond the following initial and boundary conditions:

$$u|_{t=0} = 0 ; \quad w|_{t=0} = 0 ; \quad u|_{y=\pm 1} = 0 ; \quad w|_{y=\pm 1} = 0 . \tag{2}$$

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Thus, we are to solve a system of inhomogeneous equations (1) under zero initial and boundary conditions (2). Of three equations of this system only the first and the third are interrelated and must be considered together. The second equation, with the values of u and w being known, is integrated easily and it serves to determine pressure.

Since the system of equations (1) is linear, the functions sought can be represented as a sum of certain particular and general solutions:

$$u = u_0 + \bar{u}\,, \quad w = w_0 + \bar{w}\,.$$

Here the particular solution has the form

$$\bar{u} = c_1 \cosh (y\sqrt{R}) \cos (y\sqrt{R}) + c_2 \sinh (y\sqrt{R}) \sin (y\sqrt{R}) - \frac{3}{2R}\,,$$

$$\bar{w} = c_2 \cosh (y\sqrt{R}) \cos (y\sqrt{R}) - c_1 \sinh (y\sqrt{R}) \sin (y\sqrt{R})\,,$$

where

$$c_1 = 1.5 \frac{\cosh \sqrt{R} \cos \sqrt{R}}{R (\cosh^2 \sqrt{R} - \sin^2 \sqrt{R})}\,; \quad c_2 = 1.5 \frac{\sinh \sqrt{R} \sin \sqrt{R}}{R (\cosh^2 \sqrt{R} - \sin^2 \sqrt{R})}\,; \quad R = \frac{\omega h^2}{\nu} \cos \beta\,.$$

We will write the system of homogeneous equations to determine the functions u_0 and w_0 in the following form:

$$\mathrm{Ta} \frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial y^2} - 2Rw_0\,, \quad \mathrm{Ta} \frac{\partial w_0}{\partial t} = \frac{\partial^2 w_0}{\partial y^2} + 2Ru_0\,. \tag{3}$$

We will seek its solution in the form

$$u_0 = \exp (kt)\, U(y)\,, \quad w_0 = \exp (kt)\, W(y)\,,$$

(k is a complex number). After substitution of these relations into the initial equations, we arrive at the boundary-value problem in eigenvalues for a system of ordinary differential equations:

$$\frac{d^2 U}{dy^2} = k\mathrm{Ta}U + 2RW\,, \quad \frac{d^2 W}{dy^2} = k\mathrm{Ta}W - 2RU\,, \tag{4}$$

whose solution is represented in the form

$$U = c_1 \exp (\lambda y)\,, \quad W = c_2 \exp (\lambda y)\,,$$

Here the constants λ, c_1, and c_2 are assumed to be complex numbers. After substitution of these relations into Eq. (4), we will obtain the following equation to determine λ and k:

$$(\lambda^2 - k\mathrm{Ta})^2 + 4R^2 = 0\,,$$

whence it follows that

$$k\mathrm{Ta} = \lambda^2 \pm 2Ri\,,$$

where $i = \sqrt{-1}$ is an imaginary unit.

The eigenvalues λ are found from boundary conditions (2):

$$\lambda_n = \pm n \frac{\pi}{2} \quad (n = 0, 1, 2, ...) .$$

Here the solution of system (3) has the form

$$u_n^{(1)} = \exp\left(-\frac{n^2\pi^2}{4\mathrm{Ta}} t\right) \sin\left(2t\cos\beta - n\frac{\pi}{2} y\right) ;$$

$$u_n^{(2)} = \exp\left(-\frac{n^2\pi^2}{4\mathrm{Ta}} t\right) \cos\left(2t\cos\beta - n\frac{\pi}{2} y\right) \quad (n = 0, 1, 2, ...) .$$

The structure of these functions shows that with Coriolis acceleration acting on the flow the character of the propagation of perturbations in it changes. While in the ordinary case the evolution is associated with the diffusional character of the equalization of a velocity profile, in a noninertial coordinate system velocity perturbations propagate already in the form of transverse waves that interact with one another and the coefficient of attenuation of which depends on the sole dimensionless Taylor parameter entering into this solution. As the number n increases, the coefficient of attenuation grows, while the phase velocity decreases monotonically.

Finally, the solution of system (1) can be written in the form

$$u = u_1 + u_2 + \bar{u} ; \quad w = w_1 + w_2 + \bar{w} ,$$

where

$$\begin{aligned}
u_1 &= \sum_{n=0}^{\infty} a_n \exp(-q_n t)(\cos\psi_n^+ - \cos\psi_n^-) + \sum_{n=0}^{\infty} b_n \exp(-q_n t)(\sin\psi_n^+ - \sin\psi_n^-) ;\\
w_1 &= \sum_{n=0}^{\infty} a_n \exp(-q_n t)(\sin\psi_n^+ - \sin\psi_n^-) - \sum_{n=0}^{\infty} b_n \exp(-q_n t)(\cos\psi_n^+ - \cos\psi_n^-) ;\\
u_2 &= \sum_{n=0}^{\infty} r_n \exp(-s_n t)(\cos\varphi_n^+ + \cos\varphi_n^-) + \sum_{n=0}^{\infty} p_n \exp(-s_n t)(\sin\varphi_n^+ + \sin\varphi_n^-) ;\\
w_2 &= \sum_{n=0}^{\infty} r_n \exp(-s_n t)(\sin\varphi_n^+ + \sin\varphi_n^-) + \sum_{n=0}^{\infty} p_n \exp(-s_n t)(\cos\varphi_n^+ + \cos\varphi_n^-) ;\\
q_n &= \frac{n^2\pi^2}{\mathrm{Ta}} ; \quad s_n = \frac{(2n+1)^2\pi^2}{4\mathrm{Ta}} ; \quad \psi_n^+ = 2t\cos\beta + n\pi y ; \quad \psi_n^- = 2t\cos\beta - n\pi y ;\\
\varphi_n^+ &= 2t\cos\beta + \frac{2n+1}{2}\pi y ; \quad \varphi_n^- = 2t\cos\beta - \frac{2n+1}{2}\pi y .
\end{aligned} \tag{5}$$

Since the functions sought must satisfy initial conditions (2), for $t = 0$ Eq. (5) yields

$$\sum_{n=0}^{\infty} b_n \sin(n\pi y) + \sum_{n=0}^{\infty} r_n \cos\left(\frac{2n+1}{2}\pi y\right) + 0.5\bar{u} = 0 ;$$

$$\sum_{n=0}^{\infty} a_n \sin(n\pi y) - \sum_{n=0}^{\infty} p_n \cos\left(\frac{2n+1}{2}\pi y\right) + 0.5\bar{w} = 0 ,$$

whence we obtain an expression for the constant expansion factors sought:

$$a_n = 0 ; \quad b_n = 0 ; \quad p_n = 0.5(c_2 A_n - c_1 B_n) ; \quad r_n = \frac{3}{4R} D_n - 0.5(c_1 A_n + c_2 B_n) ;$$

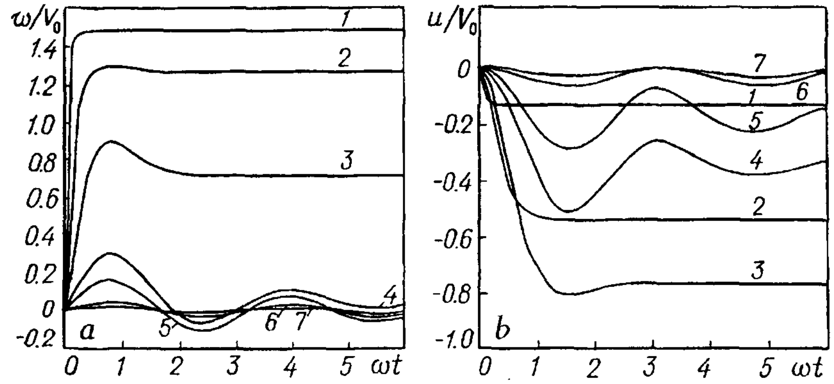

Fig. 1. Change in time of the longitudinal (a) and transverse (b) velocity components at the center of a channel for different values of the Taylor number (Ta): 1) Ta = 0; 2) 0.5; 3) 1.24; 4) 5; 5) 10; 6) 50; 7) 100.

$$A_n = \frac{(-1)^n}{R + (f_n^+)^2} (f_n^+ \cosh \sqrt{R} \cos \sqrt{R} - \sqrt{R} \sinh \sqrt{R} \sin \sqrt{R}) +$$

$$+ \frac{(-1)^n}{R + (f_n^-)^2} (\sqrt{R} \sinh \sqrt{R} \sin \sqrt{R} - \psi_n^- \cosh \sqrt{R} \cos \sqrt{R}) ;$$

$$B_n = \frac{(-1)^n}{R + (f_n^+)^2} (\sqrt{R} \cosh \sqrt{R} \cos \sqrt{R} + \psi_n^+ \sinh \sqrt{R} \sin \sqrt{R}) -$$

$$- \frac{(-1)^n}{R + (f_n^-)^2} (f_n^- \sinh \sqrt{R} \sin \sqrt{R} + \sqrt{R} \cosh \sqrt{R} \cos \sqrt{R}) ;$$

$$D_n = \frac{(-1)^n \cdot 4}{(2n+1)\pi} ; \; f_n^+ = \sqrt{R} + \frac{2n+1}{2} \pi ; \; f_n^- = \sqrt{R} - \frac{2n+1}{2} \pi .$$

The wave properties of the fundamental solutions noted above will necessarily influence the behavior of the functions sought. The interaction of waves no longer ensures, as in the ordinary case, a monotonic and smooth change in the unknown values in time, as confirmed by the results of calculations presented in Figs. 1 and 2. To calculate the functions u and w, we took 100 terms of the expansion series. This ensured the accuracy of velocity approximation at the initial time instant 10^{-6}.

Figure 1a presents the dimensionless velocity component vs time at the center of the channel ($y = 0$) for different values of the Taylor number within 0.1–100. It is seen from the figure that as the value of Ta increases, the time during which the flow reaches a steady state increases, the velocity of liquid decreases monotonically, and the flow acquires an increasingly pronounced pulsating nature. Moreover, the frequency of pulsations of the flow does not change, and it is equal to the doubled frequency of the angular velocity of rotation of the coordinate system, while the coefficient of attenuation decreases with increase in the Taylor number. The transition from a smooth monotonic change in the velocity to a pulsating mode of flow lies within the range of Taylor numbers 4–5. When $t \to \infty$, the oscillations decay, and the flow of liquid reaches the regime of steady motion. Attention is drawn to the fact that when Ta > 1, a maximum appears, at which the velocity is larger than its stationary asymptotic value.

The graphs of change in the transverse velocity component at the channel center for various Taylor numbers as functions of time are shown in Fig. 1b. The behavior of this velocity component differs from that considered above only in the fact that its oscillations lag behind the oscillations of the longitudinal component by a quarter of

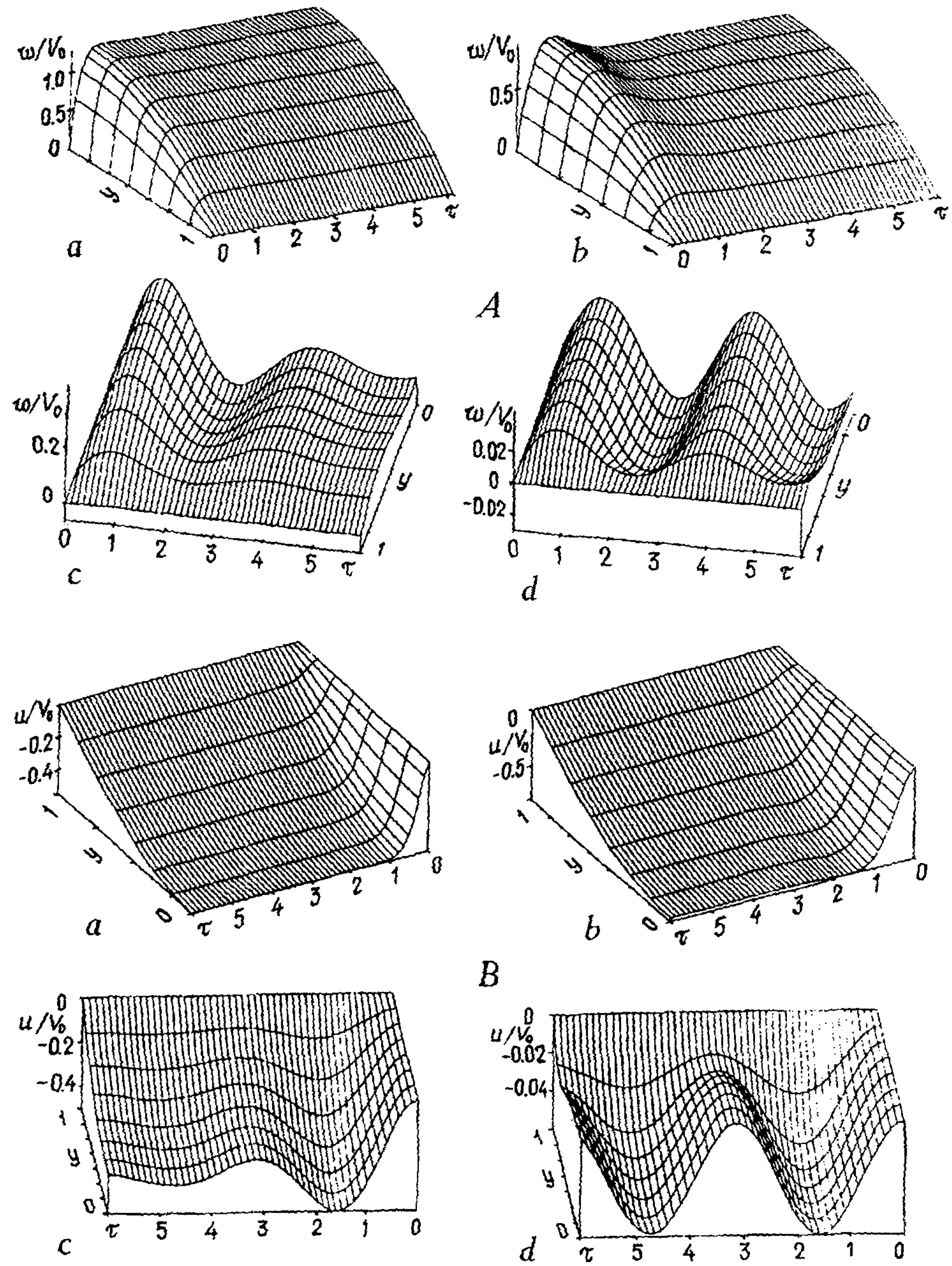

Fig. 2. Change in time of the profile of the longitudinal (*A*) and transverse (*B*) velocity components for different values of the Taylor number (Ta): a) Ta = 0.5; b) 1.24; c) 5; d) 50.

a period, and with an increase in Ta the flow velocity first increases, reaches a maximum at Ta $\approx$ 1.24, and thereafter decreases monotonically.

The mere form of the longitudinal and transverse velocity profiles and their change in time for different values of Ta are given in Fig. 2. From the graphs presented it is seen that all the points of each profile oscillate simultaneously, and their behavior in time is similar.

The above-described dynamics of the change in the character of liquid flow with Taylor number is quite explainable physically. Since the Taylor number characterizes the ratio between the Coriolis forces and the forces of viscosity, then in the flow at small values of Ta the viscous friction dominates, which precisely quenches the induced oscillations prior to the moment of their onset. As the values of Ta increase, the effect of Coriolis forces on the flow increases. Decaying aperiodic oscillations begin to appear in the system, which develop into a slowly decaying motion of the medium at large values of Ta.

In conclusion, using the data calculated, we will evaluate the possible effect of the rotation of the Earth on the appearance of nonstationary oscillations in a liquid flow. Since this mode of flow becomes noticeable only when Ta $\geq$ 5, for their appearance, say in a water flow, the half-width of the channel must be equal to 0.26 m. As we see, in many cases this value is commensurable with the transverse dimensions of many main pipelines and channels used in practice.

NOTATION

$\mathrm{Eu} = P/\rho V_0^2$, Euler number; $\mathrm{Re} = V_0 h/\nu$, Reynolds number; $\mathrm{Ta} = \omega h^2/\nu$, Taylor number; h, half-width of the channel; V_0, mean volumetric velocity of liquid flow at Ta = 0; ν, kinematic coefficient of viscosity; ρ, density; u, w, projections of the dimensionless velocity on the OX and OZ axes of a Cartesian coordinate system related to the mean volumetric velocity V_0; ω, cos α, cos β, and cos γ, the absolute value and direction cosines of the vector of the angular velocity of the rotation of the coordinate system; $\tau = \omega t$, dimensionless time; y, z, Cartesian coordinates of the point.

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